

Role of Initial Data in Higher Dimensional Quasi-Spherical Gravitational Collapse

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We study the gravitational collapse in $(n+2)$ -D quasi-spherical Szekeres space-time (which possess no killing vectors) with dust as the matter distribution. Instead of choosing the radial coordinate ‘ r ’ as the initial value for the scale factor R , we consider a power function of r as the initial scale for the radius R . We examine the influence of initial data on the formation of singularity in gravitational collapse.

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Over the last two decades gravitational collapse in spherical space-time (TBL model) has been studied extensively with dust as the matter content [1-10]. A general conclusion from these studies is that a central curvature singularity may be naked but its local or global visibility depends on the choice of initial data. If the regular initial density profile falls off rapidly having a maximum value at the centre then it is not possible to have naked singularity above five dimensional space-time [8-10].

Also from the recent past, attention has been given to study non-spherical collapse [11-20]. Most of these studies deal collapse numerically [12-15] with a few analytical works [16] (for quasi-spherical gravitational collapse, see ref. [17-20]). These are mainly concerned with special shape of the gravitating body. The present work examines the role of initial data in the formation of gravitational collapse in $(n+2)$ -D Szekeres space-time.

The metric ansatz for $(n+2)$ D Szekeres space-time is [21, 22]

$$ds^2 = dt^2 - e^{2\alpha} dr^2 - e^{2\beta} \sum_{i=1}^n dx_i^2 \quad (1)$$

where α and β are functions of all the $(n+2)$ space-time co-ordinates. But if we assume that $\beta' \neq 0$ then for inhomogeneous dust model the solution is [22]

$$e^\beta = R(t, r) e^{\nu(r, x_1, \dots, x_n)} \quad (2)$$

$$e^\alpha = \frac{R' + R \nu'}{\sqrt{1 + f(r)}} \quad (3)$$

$$e^{-\nu} = A(r) \sum_{i=1}^n x_i^2 + \sum_{i=1}^n B_i(r) x_i + C(r) \quad (4)$$

$$\dot{R}^2 = f(r) + \frac{F(r)}{R^{n-1}} \quad (5)$$

$$\rho(t, r, x_1, \dots, x_n) = \frac{n}{2} \frac{F' + (n+1)F\nu'}{R^n(R' + R\nu')} \quad (6)$$

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$$\sum_{i=1}^n B_i^2 - 4AC = -1 \quad (7)$$

where $A(r)$, $B_i(r)$ and $C(r)$ are arbitrary functions of r satisfying (7) and also in the expression (5), $f(r)$ and $F(r)$ are arbitrary functions of r alone.

A shell focusing singularity on a shell of dust occurs when it collapses at or starts expanding from the centre of matter distribution. We shall consider only the central shell focusing singularity (i.e., $R = 0$ or $\beta = -\infty$) for marginally bound case only (i.e., $f(r) = 0$). Suppose $t = t_i$ be the initial hypersurface from which the collapse develops. For initial data we assume that $R(t_i, r)$ is a monotone increasing function of r . So without any loss of generality, it is possible to make an arbitrary relabeling of the dust shells by $r \rightarrow g(r)$ such that we can choose

$$R(t_i, r) = R_0 r^k, \quad (R_0 > 0, \quad k \text{ are constants}) \quad (8)$$

Hence solving equation (5) using the initial condition (8) we get

$$R = \left[R_0^{\frac{n+1}{2}} r^{\frac{(n+1)k}{2}} - \frac{n+1}{2} \sqrt{F(r)} (t - t_i) \right]^{\frac{2}{n+1}} \quad (9)$$

The regularity of the metric co-efficients on the initial hypersurface restricts $k \geq 1$. Further for the regularity of the initial density profile

$$\rho_i(r, x_1, \dots, x_n) = \rho(t_i, r, x_1, \dots, x_n) = \frac{n}{2} \frac{F' + (n+1)F\nu'}{R_0^{n+1} r^{(n+1)k-1} (k+r\nu')} \quad (10)$$

we can write the following series expansion for $\rho_i(r)$, $F(r)$ and $\nu'(r)$ near $r = 0$ as [22]

$$\rho_i(r) = \sum_{j=0}^{\infty} \rho_j r^j, \quad (11)$$

$$F(r) = \sum_{j=0}^{\infty} F_j r^{(n+1)k+j} \quad (12)$$

and

$$\nu'(r) = \sum_{j=-1}^{\infty} \nu_j r^j, \quad (\nu_{-1} + k \geq 0) \quad (13)$$

Also using equations (11)-(13) in (10) we have the following relations among the different co-efficients

$$\begin{aligned} \rho_0 &= \frac{n(n+1)}{2} F_0 R_0^{-(n+1)}, \quad \rho_1 = \frac{n}{2} \left(n+1 + \frac{1}{k+\nu_{-1}} \right) F_1 R_0^{-(n+1)}, \\ \rho_2 &= \frac{n}{2} \left[\left(n+1 + \frac{2}{k+\nu_{-1}} \right) F_2 - \frac{F_1 \nu_0}{(k+\nu_{-1})^2} \right] R_0^{-(n+1)}, \\ \rho_3 &= \frac{n}{2} \left[\left(n+1 + \frac{3}{k+\nu_{-1}} \right) F_3 - \frac{2F_2 \nu_0}{(k+\nu_{-1})^2} - \frac{(k+\nu_{-1})\nu_1 - \nu_0^2}{(k+\nu_{-1})^3} F_1 \right] R_0^{-(n+1)}, \end{aligned} \quad (14)$$

...,

OR

$$\begin{aligned}\rho_0 &= \frac{n}{2} \left[\frac{F_1}{\nu_0} + (n+1)F_0 \right] R_0^{-(n+1)}, \quad \rho_1 = \frac{n}{2} \left[\frac{2F_2}{\nu_0} + \left\{ (n+1) - \frac{\nu_1}{\nu_0^2} \right\} F_1 \right] R_0^{-(n+1)}, \\ \rho_2 &= \frac{n}{2} \left[\frac{3F_3}{\nu_0} + \left\{ (n+1) - \frac{2\nu_1}{\nu_0^2} \right\} F_2 + \left(\frac{\nu_1^2}{\nu_0^3} - \frac{\nu_2}{\nu_0^2} \right) F_1 \right] R_0^{-(n+1)}, \\ \rho_3 &= \frac{n}{2} \left[\frac{4F_4}{\nu_0} + \left\{ (n+1) - \frac{3\nu_1}{\nu_0^2} \right\} F_3 + 2 \left(\frac{\nu_1^2}{\nu_0^3} - \frac{\nu_2}{\nu_0^2} \right) F_2 + \left(\frac{2\nu_1\nu_2}{\nu_0^3} - \frac{\nu_3}{\nu_0^2} - \frac{\nu_1^3}{\nu_0^4} \right) F_1 \right] R_0^{-(n+1)},\end{aligned}\tag{15}$$

according as $\nu_{-1} > -k$ or $\nu_{-1} = -k$.

The singularity curve $t = t_s(r)$ for the shell focusing singularity is characterized by

$$R(t_s(r), r) = 0 \quad (16)$$

So the time for central shell focusing singularity is given by

$$t_0 = t_s(0) = t_i + \frac{2R_0^{\frac{n+1}{2}}}{(n+1)\sqrt{F_0}} \quad (17)$$

Further, if $t_{ah}(r)$ is the instant for the formation of apparent horizon then we have

$$R^{n-1}(t_{ah}(r), r) = F(r) \quad (18)$$

which gives

$$t_{ah}(r) - t_0 = -\frac{R_0^{\frac{n+1}{2}}}{(n+1)F_0^{3/2}} \left[F_1 r + \left(F_2 - \frac{3F_1^2}{4F_0} \right) r^2 + \dots \right] \\ - \frac{2}{n+1} F_0^{\frac{1}{n-1}} \left[r^{\frac{(n+1)k}{n-1}} + \frac{1}{n-1} \frac{F_1}{F_0} r^{\frac{(n+1)k}{n-1}+1} + \dots \right] \quad (19)$$

The above expression shows the time difference between the formation of trapped surface at a distance r and the time of singularity formation at $r = 0$ (central singularity). Hence the necessary condition that an observer at a distance r will observe the central singularity (at least locally) is $t_{ah}(r) > t_0$ (for details see Ref. [6–9, 24]).

From physical consideration it is reasonable to assume that the initial density $\rho_i(r)$ is maximum at the centre $r = 0$. This implies that the first non-vanishing term after ρ_0 in the series expansion (see eq. (11)) for $\rho_i(r)$ should be negative. Further, one may assume that $\rho'_i(r)$ should vanish at $r = 0$ but is negative in the neighbouring region or more generally, it may be assumed that $\rho'_i(r) = \rho''_i(r) = \dots = \rho_i^{s-1}(r) = 0$ and $\rho_i^s(r) < 0$ ($s \geq 2$). Now using the relations (14) or (15) among the co-efficients we have

I. when $\nu_{-1} > -k$:

Here $\rho_1 < 0$ implies $F_1 < 0$. Also in general, $\rho_1 = \rho_2 = \dots = \rho_{s-1} = 0$ and $\rho_s < 0$ implies $F_1 = F_2 = \dots = F_{s-1} = 0$ and $F_s < 0$ with $s \geq 2$.

II. when $\nu_{-1} = -k$:

Here $\rho_1 < 0$ does not imply $F_1 < 0$. Also more generally, $\rho_1 = \rho_2 = \dots = \rho_{s-1} = 0$ and $\rho_s < 0$ may have one possible solution as $F_1 = F_2 = \dots = F_s = 0$ and $F_{s+1} < 0$ with $s \geq 2$.

In particular, if we assume the initial density to have a maximum value at $r = 0$ and falls off rapidly near $r = 0$ then we have $\rho_1 = 0, \rho_2 < 0$ near $r = 0$. Then in case I, we have $F_1 = 0, F_2 < 0$ while in case II, we may take $F_1 = F_2 = 0, F_3 < 0$ near $r = 0$.

Therefore in the present problem we have the following possibilities for naked singularity:

(a) If $F_1 < 0$ then naked singularity may appear in any dimension (≥ 4) for $\nu_{-1} \geq -k$ and $k \geq 1$.

(b) In general if we choose $F_1 = 0, F_2 = 0, \dots, F_{i-1} = 0$ and $F_i < 0$ ($i \geq 2$) then for formation of naked singularity, ‘ n ’ is restricted by the inequality

$$2 \leq n \leq \left[\frac{i+k}{i-k} \right] \quad (20)$$

with $\max(1, \frac{i}{3}) \leq k < i, \nu_{-1} \geq -k, k \geq 1$.

Here $[x]$ stands for the greatest integer in x . From the inequality (20) we note that if ‘ k ’ is very close to ‘ i ’ (but less than i) then ‘ n ’ can take larger values than 2 i.e., naked singularity may appear in much larger dimension compare to the usual four dimension. The following table shows some possible values of n for different values of i and k (with $k < i$) from the inequality (20).

TABLE: Some possible values of n for different values of i and k .

$i \rightarrow$ $k \downarrow$	2	3	4	5
$\frac{3}{2}$	$n = 2, 3, 4, 5, 6, 7$	$n = 2, 3$	$n = 2$	—
$\frac{21}{11}$	$2 \leq n \leq 41$	$n = 2, 3, 4$	$n = 2$	—
$\frac{35}{12}$	—	$2 \leq n \leq 71$	$n = 2, 3, 4, 5, 6$	$n = 2, 3$

For $\nu_{-1} > -k$ we have from equation (14) $F_1 = 0, F_2 < 0$ and hence for naked singularity

$$2 \leq n \leq \left[\frac{2+k}{2-k} \right]$$

with $1 \leq k < 2$, while for $\nu_{-1} = -k$ we choose $F_1 = 0, F_2 = 0, F_3 < 0$ and so n will be restricted by the inequality

$$2 \leq n \leq \left[\frac{3+k}{3-k} \right]$$

with $1 \leq k < 3$.

We note that for $k = 1$ the possible values of n are 2 and 3 for $\nu_{-1} > -1$ and $n = 2$ for $\nu_{-1} = -1$ i.e., naked singularity is possible only for four and five dimensions which we have shown in earlier works (see ref. [8-10, 20]).

Now we shall examine the nature of singularity by studying the outgoing radial null geodesic (ORNG) originated from the central shell focusing singularity. Let us start with the assumption that it is possible to have one or more such geodesics and we choose the form of ORNG in power series as [23, 24]

$$t = t_0 + ar^\xi, \quad (21)$$

upto leading order near $r = 0$ in $t-r$ plane with $a > 0, \xi > 0$ as constants. Using equation (16) and (17) the singularity curve can be written as (near $r = 0$)

$$t_s(r) = t_0 - \frac{F_m R_0^{\frac{n+1}{2}}}{(n+1)F_0^{3/2}} r^m \quad (22)$$

where $m \geq 1$ is an integer and F_m is the first non-vanishing term beyond F_0 . As for naked singularity we have $t < t_s(r)$ so comparing (22) with (21) for ORNG the restrictions on the two parameters ξ and a as

$$\xi \geq m \quad \text{and} \quad a < -\frac{F_m R_0^{\frac{n+1}{2}}}{(n+1)F_0^{3/2}} \quad (23)$$

Moreover, from the metric (1) we can write for ORNG

$$\frac{dt}{dr} = R' + R \nu' \quad (24)$$

We shall now examine the feasibility of the null geodesic starting from the singularity with the above restrictions for the following two cases namely, $\xi > m$ and $\xi = m$.

When $\xi > m$ then near $r = 0$ the solution for R in (9) (choosing $t_i = 0$) simplifies to

$$R = R_0 \left(-\frac{F_m}{2F_0} \right)^{\frac{2}{n+1}} r^{\frac{2m}{n+1} + k} \quad (25)$$

Now combining (21) and (25) in equation (24) we get (upto leading order in r)

$$a \xi r^{\xi-1} = R_0 \left(\nu_{-1} + k + \frac{2m}{n+1} \right) \left(-\frac{F_m}{2F_0} \right)^{\frac{2}{n+1}} r^{\frac{2m}{n+1} + k - 1} \quad (26)$$

which implies

$$\xi = k + \frac{2m}{n+1} \quad \text{and} \quad a = \frac{(\nu_{-1} + \xi)R_0}{\xi} \left(-\frac{F_m}{2F_0} \right)^{2/(n+1)} \quad (27)$$

Now if $k < m$ then n and k are bounded by the same inequalities as in (20) (with $i = m$) for the formation of naked singularity while for $k \geq m$, there will be no upper bound of n . Furthermore, we note that from equation (27), $\xi > 0$ and $a > 0$ as $\nu_{-1} \geq -k$. Thus we have the same conclusion as before (in case (b)) and it is possible to have consistent ORNG originated from the singularity.

On the other hand for $\xi = m$, we have from equation (24) using the solutions for R and ν as before, it is possible to have naked singularity if

$$n = \frac{m+k}{m-k}, \quad \max(1, \frac{m}{3}) \leq k < m \quad (28)$$

and

$$a = -\frac{1}{m} \left(-\frac{n+1}{2} \sqrt{F_0} a - \frac{F_m}{2F_0} R_0^{\frac{n+1}{2}} \right)^{\frac{1-n}{1+n}} \left[(\nu_{-1} + m) \frac{F_m}{2F_0} R_0^{\frac{n+1}{2}} + \frac{1}{2}(n+1)(\nu_{-1} + k) \sqrt{F_0} a \right] \quad (29)$$

Now we shall examine whether the restriction (23) for ‘ a ’ is consistent with the expression ‘ a ’ in equation (29). In fact equation (29) takes the form

$$2^{\frac{2}{n+1}} b m = -[-(n+1)b - \zeta]^{\frac{1-n}{1+n}} [(\nu_{-1} + m)\zeta + (n+1)(\nu_{-1} + k)b] \quad (30)$$

with the transformation

$$a = b F_0^{\frac{1}{n-1}}, \quad F_m = \frac{\zeta F_0^{\frac{m}{2}+1}}{R_0^{\frac{n+1}{2}}}. \quad (31)$$

Since equation (30) is a real valued equation of b , so we must have

$$(n+1)b + \zeta < 0,$$

which using (31) gives us the restriction on a in equation (23). Hence the geodesic (21) will have consistent solution for ‘ a ’ and ‘ ξ ’. So the above conclusion regarding the formation of naked singularity is justified. Further, introducing the variable ‘ ϕ ’ by the relation

$$\phi = -(n+1)b - \zeta \quad (32)$$

we have seen from equation (30)

$$4k^{n+1}\phi^{n-1}(\zeta + \phi)^{n+1} = [2k\zeta - (n-1)(\nu_{-1} + k)\phi]^{n+1} \quad (33)$$

with the restriction $0 < \phi < -\zeta$ and ‘ a ’ will satisfy the inequality in equation (23).

Further we observe that for $k \geq m$, n can not have any positive integral solution and hence naked singularity is not possible. Also this case (i.e., $\xi = m$) has no analogue with our previous result by comparing $t_{ah}(r)$ with t_0 . Lastly for any fixed m , k can only take those values within the limits in equation (28) which makes n , a positive integer(≥ 2). It is to be noted that if we consider next order term in the geodesic eq. (21) then it is possible to have a family of ORNG originated from the central singularity (see Ref. [24]).

Thus the formation of naked singularity strongly depends on the nature of the initial density and also on the choice of the parameter k involved in the initial choice of the scale factor R . Therefore we conclude that formation of naked singularity strongly depends on the choice of initial data for the physical parameters as well as for geometric quantity. For

future work, it will be interesting to study the dominance of initial data for physical parameters over that for geometric quantities and vice versa for formation of naked singularity.

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References:

- [1] A. Ori and T. Piran, *Phys. Rev. Lett.* **59** 2137 (1987).
- [2] F. J. Tipler, *Phys. Lett. A* **64** 8 (1987).
- [3] F. C. Mena, R. Tavakol and P. S. Joshi, *Phys. Rev. D* **62** 044001 (2000).
- [4] P.S. Joshi, *Global Aspects in Gravitation and Cosmology* (Oxford Univ. Press, Oxford, 1993).
- [5] P. S. Joshi and I. H. Dwivedi, *Commun. Math. Phys.* **166** 117 (1994); *Class. Quantum Grav.* **16** 41 (1999).
- [6] D. Christodoulou, *Commun. Math. Phys.* **93** 171 (1984); R. P. A. C. Newman, *Class. Quantum. Grav.* **3** 527 (1986).
- [7] P.S. Joshi, N. Dadhich and R. Maartens, *Phys. Rev. D* **65** 101501(R)(2002).
- [8] A. Banerjee, U. Debnath and S. Chakraborty, *gr-qc/0211099* (2002)(accepted in *Int. J. Mod. Phys. D*) .
- [9] U. Debnath and S. Chakraborty, *gr-qc/0211102* (2002) .
- [10] R. Goswami and P.S. Joshi, *gr-qc/02112097* (2002) .
- [11] C. Barabas, W. Israel and P. S. Letelier, *Phys. Lett. A* **160** 41 (1991); M. A. Pelath, K. P. Tod and R. M. Wald, *Class. Quantum Grav.* **15** 3917 (1998); F. Echeverria, *Phys. Rev. D* **47** 2271 (1993); T. A. Apostolatos and K. S. Thorne, *Phys. Rev. D* **46** 2435 (1992).
- [12] S. L. Shapiro and S. A. Teukolsky, *Phys. Rev. Lett.* **66** 994 (1991); *Phys. Rev. D* **45** 2006 (1992).
- [13] T. Nakamura, M. Shibata and K.I. Nakao, *Prog. Theor. Phys.* **89** 821 (1993) .
- [14] T. Harada, H. Iguchi and K.I. Nakao, *Phys. Rev. D* **58** 041502 (1998) .
- [15] H. Iguchi, T. Harada and K.I. Nakao, *Prog. Theor. Phys.* **101** 1235 (1999); *Prog. Theor. Phys.* **103** 53 (2000).
- [16] K. S. Thorne, 1972 in *Magic Without Magic*: John Archibald Wheeler, Ed. Klauder J (San Francisco: W. H. Freeman and Co.).
- [17] P. Szekeres, *Phys. Rev. D* **12** 2941 (1975).
- [18] P. S. Joshi and A. Krolak, *Class. Quantum Grav.* **13** 3069 (1996); S. S. Deshingkar, S. Jhingan and P. S. Joshi, *Gen. Rel. Grav.* **30** 1477 (1998).
- [19] S. M. C. V. Goncalves, *Class. Quantum Grav.* **18** 4517 (2001); *Phys. Rev. D* **65** 084045 (2002).
- [20] U. Debnath, S. Chakraborty and J. D. Barrow, *gr-qc/0305075* (2003).
- [21] P. Szekeres, *Commun. Math. Phys.* **41** 55 (1975).
- [22] S. Chakraborty and U. Debnath , *gr-qc/0304072* (2003).
- [23] S. Barve, T. P. Singh, C. Vaz and L. Witten, *Class. Quantum Grav.* **16** 1727 (1999).
- [24] U. Debnath and S. Chakraborty, *math-ph/0307024*.